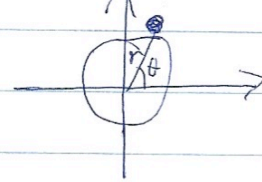
Goldstein

2.13. We first set up the general Lagrangian, then introduce a force of constraint. We build the coordinate system as usual:



$$T = \frac{1}{2} m \left[r^2 \dot{\theta}^2 + \dot{r}^2 \right], \quad V = mgr\sin\theta$$

$$L = \frac{1}{2} m \left[r^2 \dot{\theta}^2 + \dot{r}^2 \right] - mgr\sin\theta$$

$$\frac{dL}{d\dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{dL}{dr} = mr\dot{\theta}^2 - mg\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left[2r\dot{r}\dot{\theta} + mr^2\ddot{\theta}\right]$$

$$\frac{dL}{d\theta} = -mgr\cos\theta$$

$\Rightarrow$ Without external forces or constraint, the equation of motion for a particle free falling in polar coord. is given by:

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0 \\ 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\cos\theta = 0 \end{cases}$$

<u>This is completely general!</u>

The constraint will be given by a force via Lagrange multiplier.

We apply constraint $f = r - R = 0$, which introduces the Lagrange multiplier λ to give a force of constraint $Q_r = \lambda$. This ~~equation~~ force will appear only in the r equation of motion, clearly. Additionally, $f = 0 = r - R$ demands $r = R$, $\dot{r} = \ddot{r} = 0$, thus we write the new equations of motion:

$$\begin{cases} -mR\dot{\theta}^2 + mg\sin\theta = Q_r = \lambda \\ mR^2\ddot{\theta} + mgR\cos\theta = 0 \end{cases}$$

Solving for λ:

$$\frac{d\lambda}{dt} = -mR(2)\dot{\theta}\ddot{\theta} + mg\cos\theta\dot{\theta}$$

$$= \frac{d\lambda}{d\theta}\dot{\theta}$$

$$\Rightarrow \frac{d\lambda}{d\theta} = -2mR\ddot{\theta} + mg\cos\theta$$

$$= 2mg\cos\theta + mg\cos\theta = 3mg\cos\theta$$

$$\Rightarrow \lambda(\theta) = Q_r(\theta) = 3mg\sin\theta + C$$

At $\theta = \frac{\pi}{2}$, the system being in equilibrium demands that $\lambda(\theta = \frac{\pi}{2}) = mg$, thus $C = -2mg$,

$$Q(\theta) = \lambda(\theta) = 3mg\sin\theta - 2mg.$$

Solving for $Q = 0$ gives $\theta^* = \sin^{-1}\left(\frac{2}{3}\right)$

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